Multiscale correlation functions in strong turbulence

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Under the framework of Yakhot [Phys. Rev. E **57**, 1737 (1998)], we model intermittent structure functions in fully developed turbulence, based on the experimentally supported Markovian nature of turbulence cascades [Friedrich and Peinke, Phys. Rev. Lett, **78**, 863 (1997)], and calculate the multiscaling correlation functions. Fusion rules [L'vov and Procaccia, Phys. Rev. Lett. **76**, 2898 (1996)], which were experimentally tested [Benzi, Biferale, and Toschi, Phys. Rev. Lett. **80**, 3244 (1998)] to be compatible with almost uncorrelated multiplicative process are analytically checked by direct calculations.

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One of the most important issues in stationary turbulence is the intermittent behavior of velocity fluctuations in the inertial range. Understanding the statistical properties of intermittency is one of the most challenging open problems in three-dimensional fully developed turbulence. The structures that arise in a random flow of stationary turbulence resemble high peaks at random places and random times. The intervals between them are characterized by a low intensity and a large size. Rare peaks are the hallmarks of probability density fundamental's (PDF's) non-Gaussian tails. These strongly non-Gaussian activities are statistically scale-invariant processes responsible for energy transfer. Intermittency in the inertial range is usually analyzed by means of the statistical properties of velocity differences, $\delta_r u(x) = u(x+r) - u(x)$ [1]. The overwhelming majority of experimental and theoretical works have been brought forward to characterize structure functions, i.e., $S_p = \langle (\delta_r u(x))^p \rangle$. A wide agreement exists on the fact that $S_p(r)$ exhibits a scaling behavior in the limit of high Reynolds number, that is $S_p(r) \sim (r/L)^{\zeta_p}$ for $L \gg r \gg \eta_k$, where L is the scale of energy injection, $\eta_k = (\nu^3/\epsilon)^{1/4}$ is the dissipative scale, ϵ is the mean energy dissipation range, and ν is the kinematic viscosity. Rare peaks in the random flows are signaled by a nonlinear form of $\zeta(p)$. In other words the velocity increments are multifractal, and $\zeta(p)$'s do not follow the celebrated K41 theory, $\zeta(p) = p/3$. Recently [2–4] it was proposed that it would be more natural to look at single time correlations among velocity increment fluctuations at different scales,

$$\mathcal{F}_n(x|r_1, r_2, \ldots, r_n) = \langle \delta_{r_1} u(x) \delta_{r_2} u(x) \cdots \delta_{r_n} u(x) \rangle, \tag{1}$$

where all the scales r_i are lying in the inertial range, i.e., $\eta \ll r_i \ll L$. For simplicity we confine the discussion to longitudinal velocity increments. Fusion rules [2–4] that describe the asymptotic properties of n-point correlation functions when some of the coordinates tend toward one other are derived from two fundamental assumptions which are of paramount importance for a description of nonperturbative aspects of the analytic theory of stationary turbulence. The fusion rules were tested experimentally, and a good agreement between experiment and theory observed [5]. If $p < n$ pairs of coordinates of velocity differences merge, with typical separations $r_i \sim r$ for $i \leqslant p$ and the remaining separation at the order of R, such that $r \ll R \ll L$, the fused multi-scale correlation is defined as

$$\mathcal{F}_{p+q}(r,R) \equiv \langle [u(x+r) - u(x)]^p [u(x+R) - u(x)]^q \rangle$$
$$\equiv \langle [\delta_r u(x)]^p [\delta_R u(x)]^q \rangle. \tag{2}$$

It has been deduced that

$$\mathcal{F}_{p+q}(r,R) \sim S_p(r) S_{p+q}(R)/S_p(R). \tag{3}$$

On the other hand, multiscale correlation functions in high Reynolds number experimental turbulence, numerical simulations, and synthetic signal were recently investigated by Benzi *et al.* [6], and it was found that whenever a simple scaling ansatz based on uncorrelated multiplicative processes [6] is not prevented by symmetry arguments, the multiscale correlations are in good agreement with the fusion rule prediction even if strong corrections due to subleading terms are seen for small-scale separation $r/R \sim O(1)$. All the findings has led to the conclusion that multiscale correlation functions measured in turbulence are fully consistent with a multiplicative, almost uncorrelated, random process for the evolution of velocity increments in scale . Although a successful interpretation of the fusion rules can be realized by considering a multiplicative random process for the evolution of velocity increments on a length scale, it is at most a phenomenological model, and it is not based on first principles calculations. Other experimental investigations of the behavior of conditional probability densities of velocity increments in scale have shown that the Markovian nature of velocity increments in terms of length scale and in the inertial range would support the experimental data [7]. In fact the necessary condition of ''Markovianity'' for velocity increments has been tested experimentally, and from this phenomenological scenarios for modeling the intermittency have been developed [7]. The aforementioned ideas were later supported by invoking the theoretical ideas inspired by Polyakov [8] and Yakhot [9] based on the operator product expansion (OPE) and general invariances of the Navier-Stokes equation [9]. In this paper we show that, relying on Yakhot's closure for an infinite Reynolds number, the phenomenological interpretation given by Benzi *et al.* [6] in finite Reynolds numbers can still be achieved in some approximations. We will propose an underlying dynamical process in scale which incorporates the fusion rules of multiscale correlation functions in the infinite Reynolds limit. The calculations are consistent with a picture of an almost uncorrelated random multiplicative process, at least in the Fokker-Planck description of turbulence cascades. Furthermore we are able to connect the fusion rules to the Markovian nature by a simple operator formalism, by preserving all the terms in the Kramers-Moyal's evolution operator of velocity increments.

Let us start with the Navier–Stokes equation

$$\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}=\nu\nabla^2\mathbf{v}-\frac{\nabla p}{\rho}+\mathbf{f}(\mathbf{x},t),\quad \nabla\cdot\mathbf{v}=0 \tag{4}$$

for the Eulerian velocity $\mathbf{v}(\mathbf{x},t)$ and the pressure p with viscosity ν, in N-dimensions. The force $\mathbf{f}(\mathbf{x},t)$ is an external stirring force, which injects energy into the system on a length scale L. More specifically one can take, for instance a Gaussian distributed random force, which is identified by its two moments.

$$\langle f_\mu(\mathbf{x},t)f_\nu(\mathbf{x}',t')\rangle=k(0)\,\delta(t-t')k_{\mu\nu}(\mathbf{x}-\mathbf{x}') \tag{5}$$

and $\langle f_\mu(\mathbf{x},t)\rangle=0$, where $\mu,\nu=x_1,x_2,\ldots,x_N$. The correlation function $k_{\mu\nu}(r)$ is normalized to unity at the origin, and decays rapidly enough where r becomes larger than or equal to integral scale L; that is

$$k_{\mu\nu}(r_{ij})=k(0)\left[1-\frac{r_{ij}^2}{2L^2}\delta_{\mu,\nu}-\frac{(\mathbf{r}_{ij})_\mu(\mathbf{r}_{ij})_\nu}{L^2}\right],$$

with $k(0)$ and $L\equiv1$, where $r_{ij}=|\mathbf{x}_i-\mathbf{x}_j|$.

Recently Yakhot [9] generalized Polyakov's approach to Burgers turbulence [8] for strong turbulence. He used the OPE approach to close the equation for the velocity increment PDF, and showed that in homogeneous and isotropic turbulence the PDF of the longitudinal structure function $S_q=\langle[u(x+r)-u(x)]^q\rangle=\langle U^q\rangle$ satisfies the following equation in the limit $r\to0$;

$$\frac{\partial}{\partial U}U\frac{\partial P}{\partial r}-B_0\frac{\partial P}{\partial r}=-\frac{A}{r}\frac{\partial}{\partial U}UP+\frac{u_{rms}}{L}\frac{\partial^2}{\partial U^2}UP, \tag{6}$$

where $A=(3+B)/3$ and $B=-B_0>0$ and for the Navier-Stokes turbulence it has been shown that $B\sim20$ can be derived by a self-consistent calculation [9]. The last term on the right hand side is responsible for the breakdown of Galilean invariance in the limited Polyakov sense, which means that the single point u_{rms} induced by random forcing enters the resulting expression for velocity increment PDF.

Now one can show that the probability density, and as a result the conditional probability density of the velocity difference, satisfies the Kramers-Moyal (KM) evolution equations [11]

$$-\frac{\partial P}{\partial r}=L_{KM}(U,r)P,$$
$$L_{KM}=\sum_{n=1}^{\infty}(-1)^n\frac{\partial^n}{\partial U^n}[D^{(n)}(r,U)P], \tag{7}$$

where $D^{(n)}(r,U)=(\alpha_n/r)U^n+\beta_nU^{n-1}$. We have found that the coefficients α_n and β_n depend on A, B, u_{rms}, and the inertial length scale L. They are given as

$$\alpha_n=(-1)^n\frac{A}{(B+1)(B+2)(B+3)\cdots(B+n)}$$

and

$$\beta_n=(-1)^n\frac{u_{rms}}{L}\frac{1}{(B+2)(B+3)\cdots(B+n)},$$

where $\beta_1=0$ by homogeneity [11]. The coefficients $D^{(n)}(r,U)$ are the small scale limit of the conditional moments [10]. They fully characterize the statistics of eddy distribution in the inertial range, and are defined as

$$D^{(n)}(U_2)=\lim_{r_1\to r_2}\frac{1}{r_1-r_2}\int(U_1-U_2)^nP(U_1,r_1|U_2,r_2)dU_1. \tag{8}$$

It is noted that $P(U_1,r_1|U_2,r_2)$ also satisfies Eq. (7), but with a different boundary condition [10]. The Kramers-Moyal coefficients are the main observables of a Markov process from which all the terms in the Kramers-Moyal operator will be determined. It is a well known theorem (Pawula theorem) of Markov processes that whenever the fourth order Kramers-Moyal coefficient tends to zero all other terms with higher order derivatives tend to zero [10]. Then there is a distinction between Markov processes in the Fokker-Planck description, when just the first two terms in the evolution operator in scale are important, and Markov processes in which all the terms should be preserved and are encoded in the coefficients. Thanks to the detailed analysis carried over experimental data [7], the functional form of the first four Kramers-Moyal coefficients are obtained in the finite Reynolds number. It has been observed that the fourth order conditional moment tends to zero, from which, by invoking Pawula's theorem, the Fokker-Planck equation would be a reasonable evolution equation. In the meantime the present authors showed that Kramers-Moyal coefficients, which are derived from the Yakhot modeling, are consistent with the experimental observations [11]. It is interesting that the functional forms of the different coefficients $D^{(n)}(U,r)$ up to the fourth order conditional moment, identically support the experimental observations [7]. Although the functional forms of the drift and diffusion coefficients are the same, the resultant Markovian process in the framework of Yakhot's model is a Kramers-Moyal type rather than a Fokker-Planck one. However, one should be careful in comparing the phenomenological description of Friedrich and Peinke [7] and the predictions of Yakhot's model, since the phenomenological picture is grounded in observations of mostly accessible Reynolds numbers available in the experiments with free jets while Yakhot's theory [9] was developed for the infinite Reynolds number limit. Still we need much more reasonable data for making a quantitative judgment about whether the theoretical predicted Kramers-Moyal Markovian cascade from the Yakhot model is comparable with the experiment or not. The intermittency exponent of the structure functions can be derived from Eq. (7); $\zeta(p) = Ap/(B+p)$. It is easy to see that the ratio of different KM coefficients are controlled by the B parameter, as it is obvious when $B\to\infty$ K41 scaling is recovered and $B\to 0$ produces the extreme case of multiscaling related to Burgers intermittency [9]. The Kramers-Moyal's description of PDF deformation in scale was also supported by an exact computation for the compressible turbulence in the high Mach number limit [12]. In that case the numerical values of the A and B parameters are determined without any need for numerical estimation. Recalling the original idea of the Markovian property of energy cascade in scale, we take a step further and calculate the more general objects of the cascade, i.e., the unfused multiscale correlations. Assuming the Markovian nature of velocity increments in scale and the proposed form of the evolution operator $L_{KM}(U,r)$, one can in principle calculate any correlation among velocity increments in different scales:

$$\mathcal{F}_n(x|r_1,r_2,\ldots,r_n)=\langle U(r_1)U(r_2)\cdots U(r_n)\rangle$$
$$=\int dU(r_1)\cdots dU(r_n)U(r_1)\cdots U(r_n)$$
$$\times P(U_1,r_1;U_2,r_2;\ldots;U_n,r_n).$$

The joint probability $P(U_1,r_1;U_2,r_2;\ldots;U_n,r_n)$ can be calculated by taking advantage of a Markovian property in terms of conditional probabilities, i.e.,

$$P(U_1,r_1;U_2,r_2;\ldots;U_n,r_n)$$
$$=P(U_1,r_1|U_2,r_2)P(U_2,r_2|U_3,r_3)\cdots$$
$$\times P(U_{n-1},r_{n-1}|U_n,r_n)P(U_n,r_n). \tag{9}$$

The conditional PDF of velocity increments can be written as a scalar-ordered operator

$$P(U_1,\lambda_1|U_2,\lambda_2)=\mathcal{T}\left[\exp_+\left(\int_{\lambda_2}^{\lambda_1} d\lambda L_{KM}(U_1,\lambda)\right)\right]$$
$$\times\delta(U_1-U_2).$$

Thus in a calculation of n-point multiscale correlation, a series of conditional operators would emerge in the integrand of Eq. (9). When some of the coordinates coalesce, the conditional operator tends to a Dirac δ function. The reduction of the conditional probability between the coalescing coordinates simplifies the calculations. The only remaining conditional operator will be the probability of observing the typical velocity U_1 increment between one subclass of fused points, conditioned on observing the typical velocity increment U_2 in the other subclass of fused points. We explicitly examine the behavior of $\mathcal{F}_{p+q}(\lambda_1,\lambda_2)$ defined in Eq. (1), where $\lambda_1=\ln(L/r)$ and $\lambda_2=\ln(L/R)$:

$$\mathcal{F}_{p+q}(\lambda_1,\lambda_2)=\langle U^p(\lambda_1)U^q(\lambda_2)\rangle$$
$$=\int dU_1 dU_2\,\delta(U_1-U_2)\times P(U_2,\lambda_2)$$
$$\times(e^{-(\lambda_1-\lambda_2)L^\dagger_{KM}(U_1)}U_1^p)U_2^q \tag{10}$$

We restrict the calculations to the Galilean invariance (GI) invariant approximation neglecting the $O(u_{rms}r/L)$ operators in $L_{KM}(U,\lambda)$. The crucial point in the above approximation is that in the GI regime the Kramers-Moyal coefficients are scale independent, so that the entry scale dependence of the conditional probabilities would reveal a simple subtraction of the two logarithmic scales, i.e. $\lambda_1-\lambda_2$ in the exponent. Because $L^\dagger_{KM}(U_1)U_1^p=\zeta(p)U_1^p$, we will obtain the proposed form of the fusion rules in Eq. (2) with $\zeta(p)=Ap/(p+B)$. Any other multiscale correlation function is also tractable under the same approximations. The fusion rules were first introduced [2–4] by invoking two Kolmogorov type assumptions. The first one assumes scale invariance for all correlation functions in the inertial range. The second, called ''universality,'' meaning that when some arbitrary set of velocity differences in the correlation functions is fixed in a scale L, the precise choice of differences will affect the correlation functions just as an overall factor. In terms of conditional averages the second proposition means that

$$\langle U(r_1)^p|U(r_2)^q\rangle=S_p(r_1)\Phi_{p,q}(r_2), \tag{11}$$

where it is also assumed that the scale of r_2 is of the order of integral scale, while r_1 is in the inertial range. The function $\Phi_{p,q}(r_2)$ is a homogeneous function with a scaling exponent $\zeta_n-\zeta_p$, and is associated with the remaining n-p indices of $\mathcal{F}$. Mathematically the above conditional correlation is easily verified:

$$\langle U(r_1)^p|U(r_2)^q\rangle=S_p(r_1)U_2^p/S_p(r_2)$$

In Yakhot modeling the scaling hypothesis is taken into account from the very beginning, when the relevant OPE terms are chosen to close the equation governing the generating function of the longitudinal velocity increments. However, we show that at least in the framework of Yakhot modeling, the universality proposition is the *result* of the Markovianity of the evolution of velocity increments in scale. On the other hand, the necessary proof of the Markovian property was verified through the special scalar-ordered form of the conditional probabilities. This itself arose from the general invariances and scaling constraint of the Navier-Stokes equation. Thus the universality condition in the language of multiscale correlation functions has in its heart a very robust scaling invariance under as infinite parameter scaling group [1]. We should emphasise that the nonuniversal effects of the large scale motions can also manifest themselves through scale dependent terms in the Kramers-Moyal operator. Still the general form of the universality assumption would be the leading behavior, while the $O(u_{rms}r/L)$ term will be the subleading correction inducing large scale effects [13].

Within the experimentally verified approximation that neglects third and higher order KM coefficients [11,7], one can write the equivalent diffusion process on a scale which dynamically gives the relation between velocity increments at two different scales. In fact, approximating the KM equation with a Fokker-Planck evolution kernel can be interpreted as if a velocity increment U is evolved in ''scale'' λ (logarithmic length scale), by the Langevin equation [10]

$$\frac{\partial U}{\partial \lambda}=\tilde{D}^{(1)}(U,\lambda)+\sqrt{\tilde{D}^{(2)}(U,\lambda)}\,\eta(\lambda),$$

where $\eta(\lambda)$ is a white noise and the diffusion term acts as a multiplicative noise. Using Ito's prescription [10] the multipoint correlation function can be written in the form of a path integral as

$$\mathcal{F}(\lambda_1,\lambda_2)=\int \mathcal{D}U U^p(\lambda_1)U^q(\lambda_2) \times \exp\left[\int_{\lambda_1}^{\lambda_2}\left(\frac{\frac{\partial U}{\partial \lambda}-D^1(U,\lambda)}{\sqrt{D^2(U,\lambda)}}\right)d\lambda\right]^{1/2} \times P(U_2,\lambda_2) \tag{12}$$

By a simple application of Bayesian rule probability density in the outer scale, λ_2 can also be written as a path integral entering the information of a nearly Gaussian PDF on an integral scale [9]. Building up all the terms in a descriptive way, the joint probability $P(U_1,\lambda_1;U_2,\lambda_2)$ is represented as a path integral over all possible *paths* between $U(\lambda_1)$ and $U(\lambda_2)$, transferring all the information about of the integral scale into the calculation in an intermittent way. Without a further attempt at calculating the multiscale correlation by the path integral representation, we turn our attention to the Langevin dynamics instead. The resulting process is the well known Kubo [10] oscillator multiplicative process. By using the Ito [10] prescription, one can deduce that

$$\delta_{\lambda_1}U(x)=\mathcal{W}(\lambda_1,\lambda_2)\,\delta_{\lambda_2}U(x). \tag{13}$$

The multiplier $\mathcal{W}(\lambda_1,\lambda_2)$ can be easily derived in terms of α_1 and α_2 and the Wiener process at two logarithmic scales as.

$$\mathcal{W}(\lambda_1,\lambda_2)=\exp(\{-\alpha_1(\lambda_1-\lambda_2)+\sqrt{\alpha_2}[W(\lambda_1)-W(\lambda_2)]\})^{1/2}.$$

Equation (13) encodes a simple cascade process. Cascade processes are simple and well known useful tools to describe the leading phenomenology of intermittent energy transfer in the inertial range. Both anomalous scaling exponents and viscous effects [1] can be reproduced by choosing a suitable random process for the multiplier. Cascade models, not related to the equations of motion, give quantitatively correct values of ξ_{2n}; however, no model was able to address the problem of the asymmetry of the probability density function $P(U,r)\neq P(-U,r)$, and as a consequence predict the scaling exponents and amplitudes of the odd order structure functions. Relying on the derived KM equation from the Navier-Stokes equation in the infinite Reynolds numbers, we have shown that an equivalent cascade model can be related to the Fokker-Planck approximation. The approximate process corresponds to an almost uncorrelated multiplicative process over the cascade of velocity increments in scale. This is equivalent to a log-normal description of scaling exponents. Structure functions are described in terms of a multiplier $W(\lambda_1,\lambda_2)$ through $S_p(r)=C_p\langle[\mathcal{W}(r/L)]^p\rangle$, where from the Langevin equation a pure power law arises in the high Reynolds regime $\langle[\mathcal{W}(r/L)]^p\rangle\sim(r/L)^{\zeta(p)}$. In this approximation the scaling exponents would be $\zeta(p)=-p\alpha_1+p(p-1)\alpha_2/2$. From a direct calculation of the Langevin equation one can easily find the behavior of the multiscale correlation function $\mathcal{F}_{p+q}(r,R)$. In the same framework, it is straightforward to show that

$$\begin{aligned}\mathcal{F}_{p+q}(r,R)&\sim\langle[\mathcal{W}(r,R)]^p[\mathcal{W}(R,L)]^q\rangle\\&\sim\left\langle\left[\mathcal{W}\left(\frac{r}{R}\right)\right]^p\right\rangle\left\langle\left[\mathcal{W}\left(\frac{R}{L}\right)\right]^q\right\rangle\\&\sim S_p(r)S_{p+q}(R)/S_p(R).\end{aligned} \tag{14}$$

The independence of multipliers in two different scales is always assumed for the underlying cascade process; otherwise the following relation would not be held. The present framework equivalently encodes the following requirement by the obvious independency of increments in a Wiener process. Recently Benzi *et al.* [6] analyzed multiscale correlation functions from finite but highest reachable Reynolds experiments and synthetic signals. They elegantly sought to find whether fusion rules (3) are compatible with random cascade phenomenology. Their main result was that all multiscale correlation functions are well reproduced in their leading term $r/R\rightarrow 0$ by a simple uncorrelated random cascade. In Yakhot modeling of the dynamics of the longitudinal velocity increments in scale, all the above results are recovered in the Fokker-Planck approximation. The prediction of Yakhot theory for infinite Reynolds number turbulence is consistent with fusion rules; however the almost uncorrelated multiplicative process gives the statistics of multiscale correlations only in Fokker-Planck approximation. Thus, qualitatively, the theoretical predictions of Yakhot and Benzi's observations are consistent, but since there are no data available for infinite Reynolds numbers we cannot reveal anything quantitative regarding the compatibility of theory and experiment. In addition, the question of a transition to an infinite Reynolds limit cannot be answered from the theoretical modeling of Yakhot, since the theory does not have any controlling parameter. Actually the proposed closure for the dissipation anomaly is written in the infinite Reynolds limit, and seeking the transition to a finite Reynolds numbers is quantitatively impossible within that theory. It is also interesting to seek the limiting behavior of the multiscaling correlation function for Burgers turbulence, which is tractable by taking the limit of $B\rightarrow 0$ in our formulation. Equation (3) shows that the multiscaling correlation function will be independent of the outer scale R, which is consistent with our knowledge about Burgers turbulence [14]. We think that preserving all the terms in the KM equation would provide complete information about cascade in length scale, and this would answer the question of whether there are other subleading processes acting for energy transfer from large to small scales. Preserving the GI breaking terms in the corresponding stochastic processes would also answer an important unanswered question regarding the effect of uneven PDF's of velocity increments on the cascade.

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